

# COMPLETE SELF-SHRINKERS OF THE MEAN CURVATURE FLOW\*

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**ABSTRACT.** It is our purpose to study complete self-shrinkers in Euclidean space. By introducing a generalized maximum principle for  $\mathcal{L}$ -operator, we give estimates on supremum and infimum of the squared norm of the second fundamental form of self-shrinkers without assumption on *polynomial volume growth*, which is assumed in Cao and Li [5]. Thus, we can obtain the rigidity theorems on complete self-shrinkers without assumption on *polynomial volume growth*. For complete proper self-shrinkers of dimension 2 and 3, we give a classification of them under assumption of constant squared norm of the second fundamental form.

## 1. INTRODUCTION

The mean curvature flow is a well known geometric evolution equation. The study of the mean curvature from the perspective of partial differential equations commenced with Huisken's paper [14] on the flow of convex hypersurfaces. Now the study of the mean curvature flow of submanifolds of higher codimension has started to receive attentions.

One of the most important problems in the mean curvature flow is to understand the possible singularities that the flow goes through. Singularities are unavoidable as the flow contracts any closed embedded submanifold in Euclidean space eventually leading to extinction of the evolving submanifold. A key starting point for singularity analysis is Huisken's monotonicity formula because the monotonicity implies that the flow is asymptotically self-similar near a given singularity and thus, is modeled by self-shrinking solutions of the flow.

Let  $X : M^n \rightarrow \mathbb{R}^{n+p}$  be an  $n$ -dimensional submanifold in the  $n + p$ -dimensional Euclidean space  $\mathbb{R}^{n+p}$ . If the position vector  $X$  evolves in the direction of the mean curvature  $H$ , then it gives rise to a solution to the mean curvature flow:

$$F(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+p}$$

satisfying  $F(\cdot, 0) = X(\cdot)$  and

$$(1.1) \quad \frac{\partial F(p, t)}{\partial t} = H(p, t), \quad (p, t) \in M \times [0, T),$$

where  $H(p, t)$  denotes the mean curvature vector of submanifold  $M_t = F(M^n, t)$  at point  $F(p, t)$ . The equation (1.1) is called the mean curvature flow equation. A

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submanifold  $X : M^n \rightarrow \mathbb{R}^{n+p}$  is said to be a self-shrinker in  $\mathbb{R}^{n+p}$  if it satisfies

$$(1.2) \quad H = -X^N,$$

where  $X^N$  denotes the orthogonal projection of  $X$  into the normal bundle of  $M^n$  (cf. Ecker-Huisken [13]).

U. Abresch and J. Langer [1] gave a complete classification of all self-shrinkers of dimension one, that is, self-shrinkers are curve. These curves are now called Abresch-Langer curves.

In the hypersurface case, Huisken [15, 16] proved a classification theorem that the only possible smooth self-shrinkers  $M^n$  in  $\mathbb{R}^{n+1}$  with non-negative mean curvature, bounded  $|A|$ , and polynomial volume growth are isometric to  $\Gamma \times \mathbb{R}^{n-1}$  or  $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$  ( $0 \leq k \leq n$ ). Here,  $\Gamma$  is a Abresch-Langer curve and  $S^k(\sqrt{k})$  is a  $k$ -dimensional sphere. Colding and Minicozzi [10] showed that Huisken's classification theorem still holds without the assumption that  $|A|$  is *bounded*. Furthermore, they showed that the only smooth embedded entropy stable self-shrinkers with polynomial volume growth in  $\mathbb{R}^{n+1}$  are the hyperplane  $\mathbb{R}^n$ , the sphere  $S^n(\sqrt{n})$  and the cylinders  $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ ,  $1 \leq m \leq n-1$ . Kleene-Møller [18] classified complete embedded self-shrinkers of revolution in  $\mathbb{R}^{n+1}$ . Based on an identity of Colding and Minicozzi (see (9.42) in [10]), Le and Sesum [20] proved a gap theorem on the squared norm of the second fundamental form for self-shrinkers of codimension one:

**Theorem A** (Le and Sesum [20]). Let  $M^n$  be an  $n$ -dimensional complete embedded self-shrinker without boundary and with polynomial volume growth in  $\mathbb{R}^{n+1}$ . If the squared norm  $|A|^2$  of the second fundamental form satisfies  $|A|^2 < 1$ , then  $M^n$  is a hyperplane.

In the higher codimension case, K. Smoczyk in [22] proved that let  $M^n$  be a complete self-shrinker with  $H \neq 0$  and with parallel principal normal vector  $\nu = H/|H|$  in the normal bundle, if  $M^n$  has uniformly bounded geometry, then  $M^n$  must be  $\Gamma \times \mathbb{R}^{n-1}$  or  $\tilde{M}^r \times \mathbb{R}^{n-r}$ . Here  $\Gamma$  is an Abresch-Langer curve and  $\tilde{M}$  is a minimal submanifold in sphere. Very recently, Li and Wei [21] have proved this result in a weaker condition. Furthermore, Cao and Li [5] extended the classification theorem for self-shrinkers in Le and Sesum [20] to arbitrary codimension, and proved the following

**Theorem B** (Cao and Li [5]). Let  $M^n$  be an  $n$ -dimensional complete self-shrinker without boundary and with polynomial volume growth in  $\mathbb{R}^{n+p}$  ( $p \geq 1$ ). If the squared norm  $|A|^2$  of the second fundamental form satisfies  $|A|^2 \leq 1$ , then  $M^n$  is one of the followings:

- (1) a round sphere  $S^n(\sqrt{n})$  in  $\mathbb{R}^{n+1}$ ,
- (2) a cylinder  $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ ,  $1 \leq m \leq n-1$  in  $\mathbb{R}^{n+1}$ ,
- (3) a hyperplane in  $\mathbb{R}^{n+1}$ .

We should remark that, in proofs of the above theorems for complete and non-compact self-shrinkers, integral formulas are exploited as a main method. In order to guarantee that the integration by part holds, the condition of *polynomial volume growth* plays a very important role. Moreover, Cao and Li [5] have asked whether it is possible to remove the assumption on *polynomial volume growth* in their theorem.

In this paper, our purpose is to study complete self-shrinkers without the assumption on *polynomial volume growth*. In order to do it, we extend the generalized maximum principle of Yau to  $\mathcal{L}$ -operator (see Theorem 3.1). By making use of the generalized maximum principle for  $\mathcal{L}$ -operator, we prove the following:

**Theorem 1.1.** *Let  $X : M^n \rightarrow \mathbb{R}^{n+p}$  ( $p \geq 1$ ) be an  $n$ -dimensional complete self-shrinker without boundary in  $\mathbb{R}^{n+p}$ , then one of the following holds:*

- (1)  $\sup |A| \geq 1$ ,
- (2)  $|A| \equiv 0$ , i.e.  $M^n$  is a hyperplane in  $\mathbb{R}^{n+1}$ .

**Corollary 1.1.** *Let  $X : M^n \rightarrow \mathbb{R}^{n+p}$  ( $p \geq 1$ ) be a complete self-shrinker without boundary, and satisfy*

$$\sup |A|^2 < 1.$$

*Then  $M$  is a hyperplane in  $\mathbb{R}^{n+1}$ .*

**Remark 1.1.** *The round sphere  $S^n(\sqrt{n})$  and the cylinder  $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ ,  $1 \leq k \leq n-1$  are complete self-shrinkers in  $\mathbb{R}^{n+1}$  with  $|A| = 1$ . Thus, our result is sharp.*

**Theorem 1.2.** *Let  $X : M^n \rightarrow \mathbb{R}^{n+1}$  be a complete self-shrinker without boundary. If  $\inf H^2 > 0$  and  $|A|^2$  is bounded, then  $\inf |A|^2 \leq 1$ .*

**Corollary 1.2.** *Let  $X : M^n \rightarrow \mathbb{R}^{n+1}$  be a complete self-shrinker without boundary. If  $\inf H^2 > 0$  and  $|A|^2$  is constant, then  $|A|^2 \equiv 1$  and  $M^n$  is the round sphere  $S^n(\sqrt{n})$  or the cylinder  $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ ,  $1 \leq k \leq n-1$ .*

**Remark 1.2.** *In [5, 10, 15, 16] and so on, they assume that  $M^n$  has polynomial volume growth. In our results, we do not assume the condition on polynomial volume growth. We should notice that condition  $\inf H^2 > 0$  is necessary since Angenent [2] has proved that there exist embedded self-shrinkers from  $S^1 \times S^{n-1}$  into  $\mathbb{R}^{n+1}$  with  $\inf H^2 = 0$  (cf. [18]).*

In section 4, we shall consider complete proper self-shrinkers of 2 and 3 dimensions. We will try to classify complete proper self-shrinkers of 2 and 3 dimensions under condition that the squared norm of the second fundamental form is constant.

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## 2. PRELIMINARIES

Let  $X : M^n \rightarrow \mathbb{R}^{n+p}$  be an  $n$ -dimensional connected submanifold of the  $(n+p)$ -dimensional Euclidean space  $\mathbb{R}^{n+p}$ . We choose a local orthonormal frame field  $\{e_A\}_{A=1}^{n+p}$  in  $\mathbb{R}^{n+p}$  with dual coframe field  $\{\omega_A\}_{A=1}^{n+p}$ , such that, restricted to  $M^n$ ,  $e_1, \dots, e_n$  are tangent to  $M^n$ . The following conventions on the ranges of indices are used in this paper:

$$1 \leq A, B, C, D \leq n+p, \quad 1 \leq i, j, k, l \leq n, \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

Then we have

$$dX = \sum_i \omega_i e_i, \quad de_i = \sum_j \omega_{ij} e_j + \sum_\alpha \omega_{i\alpha} e_\alpha$$

and

$$de_\alpha = \sum_i \omega_{\alpha i} e_i + \sum_\beta \omega_{\alpha\beta} e_\beta.$$

We restrict these forms to  $M^n$ , then

$$(2.1) \quad \omega_\alpha = 0 \quad \text{for} \quad n+1 \leq \alpha \leq n+p$$

and the induced Riemannian metric of  $M^n$  is written as  $ds_M^2 = \sum_i \omega_i^2$ . From (2.1)

and Cartan's lemma, we get

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

The induced structure equations of  $M^n$  are given by

$$\begin{aligned} d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji}, \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \end{aligned}$$

where

$$(2.2) \quad R_{ijkl} = \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha)$$

denotes components of the curvature tensor of  $M^n$ . The second fundamental form and the mean curvature vector field of  $M^n$  are given by

$$A = \sum_{\alpha, i, j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$$

and

$$\mathbf{H} = \sum_\alpha H^\alpha e_\alpha = \sum_\alpha \sum_i h_{ii}^\alpha e_\alpha,$$

respectively. Let  $|A|^2 = \sum_{\alpha, i, j} (h_{ij}^\alpha)^2$  be the squared norm of the second fundamental form and  $H = |\mathbf{H}|$  denote the mean curvature of  $M^n$ . From (2.2), components of the Ricci curvature of  $M^n$  are given by

$$(2.3) \quad R_{ik} = \sum_\alpha H^\alpha h_{ik}^\alpha - \sum_{\alpha, j} h_{ij}^\alpha h_{jk}^\alpha.$$

Let  $R_{\alpha\beta ij}$  denote components of the normal curvature tensor in the normal bundle. We have Ricci equations:

$$(2.4) \quad R_{\alpha\beta kl} = \sum_i \left( h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta \right).$$

Defining the covariant derivative of  $h_{ij}^\alpha$  by

$$(2.5) \quad \sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{ik}^\alpha \omega_{kj} + \sum_k h_{kj}^\alpha \omega_{ki} + \sum_\beta h_{ij}^\beta \omega_{\beta\alpha},$$

we obtain the Codazzi equations

$$(2.6) \quad h_{ijk}^\alpha = h_{ikj}^\alpha.$$

By taking exterior differentiation of (2.5), and defining

$$(2.7) \quad \sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum_l h_{ljk}^\alpha \omega_{li} + \sum_l h_{ilk}^\alpha \omega_{lj} + \sum_l h_{ijl}^\alpha \omega_{lk} + \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha},$$

we have the following Ricci identities:

$$(2.8) \quad h_{ijk}^\alpha - h_{ijlk}^\alpha = \sum_m h_{mj}^\alpha R_{mikl} + \sum_m h_{im}^\alpha R_{mjkl} + \sum_\beta h_{ij}^\beta R_{\beta\alpha kl}.$$

Let  $f$  be a smooth function on  $M^n$ , we define the covariant derivatives  $f_i$ ,  $f_{ij}$ , and the Laplacian of  $f$  as follows

$$df = \sum_i f_i \omega_i, \quad \sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji}, \quad \Delta f = \sum_i f_{ii}.$$

The first and second covariant derivatives of the mean curvature vector field  $\mathbf{H}$  are defined by

$$\begin{aligned} \sum_i H_{,i}^\alpha \omega_i &= dH^\alpha + \sum_\beta H^\beta \omega_{\beta\alpha}, \\ \sum_j H_{,ij}^\alpha \omega_j &= dH_{,i}^\alpha + \sum_j H_{,j}^\alpha \omega_{ji} + \sum_\beta H_{,i}^\beta \omega_{\beta\alpha}. \end{aligned}$$

The following elliptic operator  $\mathcal{L}$  introduced by Colding and Minicozzi in [10] will play a very important role in this paper:

$$(2.9) \quad \mathcal{L}f = \Delta f - \langle X, \nabla f \rangle$$

where  $\Delta$  and  $\nabla$  denote the Laplacian and the gradient operator on the self-shrinker, respectively and  $\langle \cdot, \cdot \rangle$  denotes the standard inner product of  $\mathbb{R}^{n+p}$ . In [7], we have studied eigenvalues of the  $\mathcal{L}$  operator. The sharp universal estimates for eigenvalues of the  $\mathcal{L}$  operator on compact self-shrinkers are obtained.

### 3. PROOF OF MAIN RESULTS

In order to prove our results, first of all, we prove the following generalized maximum principle for  $\mathcal{L}$ -operator on self-shrinkers:

**Theorem 3.1.** (*Generalized maximum principle for  $\mathcal{L}$ -operator*) *Let  $X : M^n \rightarrow \mathbb{R}^{n+p}$  ( $p \geq 1$ ) be a complete self-shrinker with Ricci curvature bounded from below. Let  $f$  be any  $C^2$ -function bounded from above on this self-shrinker. Then, there exists a sequence of points  $\{p_k\} \subset M^n$ , such that*

$$(3.1) \quad \lim_{k \rightarrow \infty} f(X(p_k)) = \sup f, \quad \lim_{k \rightarrow \infty} |\nabla f|(X(p_k)) = 0, \quad \limsup_{k \rightarrow \infty} \mathcal{L}f(X(p_k)) \leq 0.$$

*Proof.* Since this self-shrinker is a complete Riemannian manifold with Ricci curvature bounded from below and  $f$  is a  $C^2$ -function bounded from above on it, by

the generalized maximum principle of Yau in [8], then, there is a sequence of points  $\{p_k\} \subset M^n$ , such that

$$(3.2) \quad \lim_{k \rightarrow \infty} f(X(p_k)) = \sup f,$$

$$\lim_{k \rightarrow \infty} |\nabla f|(X(p_k)) = \lim_{k \rightarrow \infty} \frac{2(f(X(p_k)) - f(X(p_0)) + 1)\gamma(p_k)}{k(\gamma^2(p_k) + 2) \log(\gamma^2(p_k) + 2)} = 0,$$

and

$$(3.3) \quad \limsup_{k \rightarrow \infty} \Delta f(X(p_k)) \leq 0,$$

where  $\gamma(p)$  denotes the length of the geodesic from a fixed point  $X(p_0)$  to  $X(p)$ . Since  $X$  is the position vector, then, we have

$$|X(p_k)| \leq \gamma(p_k) + |X(p_0)|$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\langle X(p_k), \nabla f(X(p_k)) \rangle| &\leq |\nabla f(X(p_k))| \cdot |X(p_k)| \\ &= \frac{2(f(X(p_k)) - f(X(p_0)) + 1)\gamma(p_k)}{k(\gamma^2(p_k) + 2) \log(\gamma^2(p_k) + 2)} \cdot |X(p_k)| \\ &\leq \frac{2(f(X(p_k)) - f(X(p_0)) + 1)\gamma(p_k)(\gamma(p_k) + |X(p_0)|)}{k(\gamma^2(p_k) + 2) \log(\gamma^2(p_k) + 2)} \\ &\leq \frac{2(f(X(p_k)) - f(X(p_0)) + 1)}{k \log(\gamma^2(p_k) + 2)} + \frac{2(f(X(p_k)) - f(X(p_0)) + 1)\gamma(p_k)|X(p_0)|}{k(\gamma^2(p_k) + 2) \log(\gamma^2(p_k) + 2)}. \end{aligned}$$

According to (3.2) and the above inequality, we have

$$\lim_{k \rightarrow \infty} |\langle X(p_k), \nabla f(X(p_k)) \rangle| = 0.$$

Since  $\mathcal{L}f = \Delta f - \langle X, \nabla f \rangle$ , the above formula and (3.3) imply

$$\limsup_{k \rightarrow \infty} \mathcal{L}f(X(p_k)) \leq 0.$$

□

Now we prove the theorem 1.1 as follows:

*Proof of Theorem 1.1.* Since  $M^n$  is a complete self-shrinker, the self-shrinker equation (1.2) is equivalent to

$$(3.4) \quad H^\alpha = -\langle X, e_\alpha \rangle, \quad n+1 \leq \alpha \leq n+p.$$

Taking covariant derivative of (3.4) with respect to  $e_i$ , we have

$$(3.5) \quad H_{,i}^\alpha = \sum_k h_{ik}^\alpha \langle X, e_k \rangle, \quad 1 \leq i \leq n, \quad n+1 \leq \alpha \leq n+p.$$

Furthermore, by taking covariant derivative of (3.5) with respect to  $e_j$ , we have

$$(3.6) \quad \begin{aligned} H_{,ij}^\alpha &= \sum_k h_{ikj}^\alpha \langle X, e_k \rangle + h_{ij}^\alpha + \sum_{\beta, k} h_{ik}^\alpha h_{kj}^\beta \langle X, e_\beta \rangle \\ &= \sum_k h_{ikj}^\alpha \langle X, e_k \rangle + h_{ij}^\alpha - \sum_{\beta, k} H^\beta h_{ik}^\alpha h_{kj}^\beta, \end{aligned}$$

According to (3.6), we obtain

$$(3.7) \quad \mathcal{L}|H|^2 = 2|\nabla H|^2 + 2|H|^2 - 2 \sum_{\alpha, \beta, i, k} H^\alpha H^\beta h_{ik}^\alpha h_{ik}^\beta.$$

In fact,

$$\begin{aligned} \mathcal{L}|H|^2 &= \Delta|H|^2 - \langle x, \nabla|H|^2 \rangle \\ &= 2|\nabla H|^2 + 2 \sum_{\alpha, i} H^\alpha H_{,ii}^\alpha - 2 \sum_{\alpha, k} H^\alpha H_{,k}^\alpha \langle X, e_k \rangle \\ &= 2|\nabla H|^2 - 2 \sum_{\alpha, k} H^\alpha H_{,k}^\alpha \langle X, e_k \rangle \\ &\quad + 2 \sum_{\alpha} H^\alpha \left( \sum_k H_{,k}^\alpha \langle X, e_k \rangle + H^\alpha - \sum_{\beta, i, k} H^\beta h_{ik}^\alpha h_{ik}^\beta \right) \\ &= 2|\nabla H|^2 + 2|H|^2 - 2 \sum_{\alpha, \beta, i, k} H^\alpha H^\beta h_{ik}^\alpha h_{ik}^\beta. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\left| \sum_{\alpha, \beta, i, k} H^\alpha H^\beta h_{ik}^\alpha h_{ik}^\beta \right| \leq |A|^2 |H|^2.$$

Hence, from (3.7) and the above inequality, we get

$$(3.8) \quad \mathcal{L}|H|^2 \geq 2|\nabla H|^2 + 2(1 - |A|^2)|H|^2.$$

If  $\sup |A|^2 \geq 1$ , there is nothing to do. From now, we assume that  $\sup |A|^2 < 1$ . Thus,  $\sum_{\alpha, i, j} (h_{ij}^\alpha)^2 < 1$ . Together with (2.3), it is easily seen that Ricci curvature

is bounded from below. Since  $\frac{|H|^2}{n} \leq |A|^2 < 1$  and by applying the generalized maximum principle for  $\mathcal{L}$ -operator to the function  $H^2$ , we have, from (3.8)

$$0 \geq \limsup \mathcal{L}|H|^2 \geq 2(1 - \sup |A|^2) \sup |H|^2.$$

Hence, from  $\sup |A| < 1$ , we have  $\sup |H|^2 = 0$ , that is,  $H \equiv 0$ .  $M^n$  is totally geodesic. From (1.2), we know that  $M^n$  is a smooth minimal cone. Hence,  $M^n$  is a hyperplane and  $|A| \equiv 0$ . □

*Proof of Theorem 1.2.* Since  $|A|^2$  is bounded, we know that  $H$  is bounded and the Ricci curvature is bounded from below by (2.3). Without loss of generality, we can assume that  $\inf H > 0$  according to  $\inf H^2 > 0$ . By a direct computation, we have

$$\mathcal{L}H = (1 - |A|^2)H.$$

Applying the generalized maximum principle for  $\mathcal{L}$ -operator to  $-H$ , we obtain

$$0 \leq (1 - \inf |A|^2) \inf H.$$

Since  $\inf H > 0$ , we have  $\inf |A|^2 \leq 1$ . This finishes the proof of the theorem 1.2. □

*Proof of Corollary 1.2.* According to the theorem 1.2, we have  $\inf |A|^2 \leq 1$ . Since  $H \neq 0$ , we know that  $M^n$  is not totally geodesic. According to the theorem 1.1, we know  $\sup |A|^2 \geq 1$ . Since  $|A|^2$  is constant, we obtain  $|A|^2 \equiv 1$ . Since the codimension of  $M^n$  is one, we have

$$(3.9) \quad \frac{1}{2} \mathcal{L}|A|^2 = |\nabla A|^2 + |A|^2(1 - |A|^2).$$

Indeed, since

$$h_{ijk}^{n+1} = h_{kji}^{n+1} + \sum_m h_{mi}^{n+1} R_{mkjk} + \sum_m h_{km}^{n+1} R_{mijk},$$

we have

$$\begin{aligned} \Delta h_{ij}^{n+1} &= \sum_k h_{kji}^{n+1} + \sum_{m,k} h_{mi}^{n+1} R_{mkjk} + \sum_{m,k} h_{km}^{n+1} R_{mijk} \\ &= H_{,ij} + H \sum_k h_{ki}^{n+1} h_{kj}^{n+1} - |A|^2 h_{ij}^{n+1} \\ &= \sum_k h_{ikj}^{n+1} \langle X, e_k \rangle + h_{ij}^{n+1} - |A|^2 h_{ij}^{n+1}. \end{aligned}$$

Hence, we have

$$(3.10) \quad \mathcal{L} h_{ij}^{n+1} = (1 - |A|^2) h_{ij}^{n+1}.$$

From (3.10), we infer

$$\frac{1}{2} \mathcal{L}|A|^2 = |\nabla A|^2 + |A|^2(1 - |A|^2).$$

Therefore, from (3.9), we obtain  $|\nabla A|^2 \equiv 0$  since  $|A|^2 \equiv 1$ . Namely, the second fundamental form of  $M^n$  is parallel. According to the theorem of Lawson [19], we know that  $M^n$  is isometric to the round sphere  $S^n(\sqrt{n})$  or the cylinder  $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ ,  $1 \leq k \leq n-1$ .

□

#### 4. SELF-SHRINKERS OF DIMENSION TWO AND THREE

In this section, we assume that  $X : M^n \rightarrow \mathbb{R}^{n+p}$  is a complete proper self-shrinker with  $n = 2$  or  $n = 3$ . In [12], Ding and Xin have proved that a two dimensional complete proper self-shrinker in  $\mathbb{R}^3$  is a plane, a sphere or a cylinder. By the theorems of Li and Wei [21] and a simple observation, we can prove the following.

**Theorem 4.1.** *Let  $X : M^3 \rightarrow \mathbb{R}^5$  be a 3-dimensional complete proper self-shrinker without boundary and with  $H > 0$ . If the principal normal  $\nu = \frac{\mathbf{H}}{H}$  is parallel in the normal bundle of  $M^3$  and the squared norm of the second fundamental form is constant, then  $M^3$  is one of the following:*

- (1)  $S^k(\sqrt{k}) \times \mathbb{R}^{3-k}$ ,  $1 \leq k \leq 3$  with  $|A|^2 = 1$ ,
- (2)  $S^1(1) \times S^1(1) \times \mathbb{R}$  with  $|A|^2 = 2$ ,
- (3)  $S^1(1) \times S^2(\sqrt{2})$  with  $|A|^2 = 2$ ,
- (4) the three dimensional minimal isoparametric Cartan hypersurface with  $|A|^2 = 3$ .



*Proof.* Since  $M^3$  is a complete proper self-shrinker, we know that  $M^3$  has polynomial volume growth from the result of Ding and Xin [11] or X. Cheng and Zhou [9]. Thus, from the theorem 1.1 of Li and Wei [21], we know that  $M^3$  is isometric to  $\Gamma \times \mathbb{R}^2$  or  $\tilde{M}^r \times \mathbb{R}^{3-r}$ , where  $\Gamma$  is an Abresch-Langer curve and  $\tilde{M}$  is a compact minimal hypersurface in sphere  $S^{r+1}(\sqrt{r})$ .

Since  $|A|^2$  is constant, then the Abresch-Langer curve  $\Gamma$  must be a circle. In this case,  $M^3$  is isometric to  $S^1(1) \times \mathbb{R}^2$ .

If  $|A|^2 \leq 1$ , from the results of Cao and Li [5], we have  $|A|^2 = 1$  and  $M^3$  is  $S^k(\sqrt{k}) \times \mathbb{R}^{3-k}$ ,  $1 \leq k \leq 3$ . Hence, we can only consider the case of  $|A|^2 > 1$ .

When  $r = 2$ ,  $\tilde{M}$  is a compact minimal surface in sphere  $S^3(\sqrt{2})$  with the squared norm of the second fundamental form  $|\tilde{A}|^2 = |A|^2 - 1$ . Thus,  $\tilde{M}$  is the Clifford torus  $S^1(1) \times S^1(1)$  in  $S^3(\sqrt{2})$ .

When  $r = 3$ ,  $\tilde{M}$  is a compact minimal hypersurface in sphere  $S^4(\sqrt{3})$  with a constant squared norm of the second fundamental form, that is,  $|\tilde{A}|^2 = |A|^2 - 1$ . Thus,  $\tilde{M}$  is the Clifford torus  $S^1(1) \times S^2(\sqrt{2})$  in  $S^4(\sqrt{3})$  with  $|A|^2 = 2$  or the three dimensional minimal isoparametric Cartan hypersurface in  $S^4(\sqrt{3})$  with  $|A|^2 = 3$  according to the solution of Chern's conjecture for  $n = 3$  in [6]. This finishes the proof of the theorem 4.1.  $\square$

**Theorem 4.2.** *Let  $X : M^2 \rightarrow \mathbb{R}^{2+p}$  ( $p \geq 1$ ) be a 2-dimensional complete proper self-shrinker without boundary and with  $H > 0$ . If the principal normal  $\nu = \frac{\mathbf{H}}{H}$  is parallel in the normal bundle of  $M^2$  and the squared norm of the second fundamental form is constant, then  $M^2$  is one of the following:*

- (1)  $S^k(\sqrt{k}) \times \mathbb{R}^{2-k}$ ,  $1 \leq k \leq 2$  with  $|A|^2 = 1$ ,
- (2) the Boruvka sphere  $S^2(\sqrt{m(m+1)})$  in  $S^{2m}(\sqrt{2})$  with  $p = 2m - 1$  and  $|A|^2 = 2 - \frac{2}{m(m+1)}$ ,
- (3) a compact flat minimal surface in  $S^{2m+1}(\sqrt{2})$  with  $p = 2m$  and  $|A|^2 = 2$ .

*Proof.* Since  $M^2$  is a complete proper self-shrinker, we know that  $M^2$  has polynomial volume growth from the result of Ding and Xin [11] or X. Cheng and Zhou [9]. Thus, from the theorem 1.1 of Li and Wei [21], we know that  $M^2$  is isometric to  $\Gamma \times \mathbb{R}^1$  or  $\tilde{M}^2$ , where  $\Gamma$  is an Abresch-Langer curve and  $\tilde{M}$  is a compact minimal surface in sphere  $S^{p+1}(\sqrt{2})$ .

Since  $|A|^2$  is constant, then the Abresch-Langer curve  $\Gamma$  must be a circle. In this case,  $M^2$  is isometric to  $S^1(1) \times \mathbb{R}$ .

If  $|A|^2 \leq 1$ , from the results of Cao and Li [5], we have  $|A|^2 = 1$  and  $M^2$  is  $S^k(\sqrt{k}) \times \mathbb{R}^{2-k}$ ,  $1 \leq k \leq 2$ . Hence, we can only consider the case of  $|A|^2 > 1$ .

Since  $\tilde{M}$  is a compact minimal surface in sphere  $S^{p+1}(\sqrt{2})$  with a constant squared norm of the second fundamental form, that is,  $|\tilde{A}|^2 = |A|^2 - 1$ . Thus,  $\tilde{M}$  is a compact minimal surface in sphere  $S^{p+1}(\sqrt{2})$  with constant Gauss curvature. According to the classification of minimal surface in sphere  $S^{p+1}(\sqrt{2})$  with constant Gauss curvature due to Bryant [3] (cf. Calabi [4], Kenmotsu [17] and Wallach [23]), we know that  $M^2$  is isometric to a Boruvka sphere  $S^2(\sqrt{m(m+1)})$  in  $S^{2m}(\sqrt{2})$  with  $p = 2m - 1$  and  $|A|^2 = 2 - \frac{2}{m(m+1)}$  or a compact flat minimal surface in  $S^{2m+1}(\sqrt{2})$  with  $p = 2m$  and  $|A|^2 = 2$ . This finishes the proof of the theorem 4.2.  $\square$

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